SPACES OF SOLUTIONS OF RELATIVISTIC FIELD THEORIES WITH CONSTRAINTS

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Dedicated to Professor Bleuler on the occasion of his retirement.

1. Introduction

In this paper I shall explain how the reduction results of Marsden and Weinstein [38] can be used to study the space of solutions of relativistic field theories. Two of the main examples that will be discussed are the Einstein equations and the Yang-Mills equations.

The basic paper on spaces of solutions is that of Segal [49]. That paper deals with unconstrained systems and is primarily motivated by semilinear wave equations. We are mainly concerned here with systems with constraints in the sense of Dirac. Roughly speaking, these are systems whose four dimensional Euler-Lagrange equations are not all hyperbolic but rather split into hyperbolic evolution equations and elliptic constraint equations.

The methods that have been used to study these problems are of two types. First, there have been direct four dimensional attacks which, for example, put symplectic and multi-symplectic structures on the space of all solutions. These procedures are geometrically appealing since they are manifestly covariant. Since so many people have worked in this area, we merely refer the reader to [27,29,34,52,53] and references therein. Secondly, people have used the 3 + 1 or "geometrodynamic" approach. For the latter, one selects appropriate projections of the four dimensional fields on each spacelike hypersurface and imposes Hamiltonian evolution equations together with constraints. This procedure is generally called the "Dirac theory of constraints". Two

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good references are [30] and [31]. For vacuum relativity the procedure is sometimes called the "ADM formalism" after Arnowitt, Deser, Misner and Dirac (see [39]). From an analytical point of view, this second method is more powerful. It enables one to prove that spaces of solutions are a manifold at most points and to precisely investigate their symplectic structure. This paper will discuss this second method in the context of [38].

2. Some Additional Background and History.

Before embarking on a discussion of the mathematics we shall continue to review some of the background and history. This review does not pretend to be exhaustive and does omit a number of basic papers. However our intent is only to highlight some of the papers that are basic to the point of view we wish to develop.

As we have already mentioned, Segal's paper [49] gives a framework for the unconstrained theory. This leads naturally to an abstract theory of infinite dimensional Hamiltonian systems, as in [36] and [11].

The first example with constraints whose solution space was seriously studied was general relativity. In retrospect, general relativity is a harder example than Yang-Mills fields. However, developments in perturbation theory and historical circumstances dictated that relativity be done first.

2a. General Relativity

The first thing to do is to set up an infinite dimensional symplectic manifold and to realize the Einstein equations as Hamiltonian evolution equations together with constraints. An important point is that the constraints are the zero set of the conserved quantity generated by the gauge group of general relativity, i.e. the group of diffeomorphisms of spacetime. This is a fairly routine procedure given the existing ADM formalism and was carried out in [22]. (There were associated advances in the existence and uniqueness theorems; cf. [32,21,33,15] etc.).

The notation we shall use for this formalism is as follows. Let $(V, {}^{(4)}g)$ be a given spacetime. Let a slicing be given that is based on a fixed 3 manifold M. By restricting ${}^{(4)}g$ to each hypersurface in the slicing, we get a curve $g(\lambda)$ of Riemannian metrics on M. The basic symplectic space is $T^*\mathcal{M}$, the L^2-cotangent bundle of the space of Riemannian metrics on M. The conjugate variables π are symmetric tensor densities related to the extrinsic curvature (second fundamental form) of the hypersurface. The tangents to the parameter lines of the slicing decompose into normal and tangential parts determining the lapse function N and the shift vector field X of the slicing. The choice of a particular slicing is basically a choice of gauge.

Einstein's vacuum equations $\mathrm{Ein}({}^{(4)}g) = 0$ (the Einstein tensor formed from ${}^{(4)}g$) are equivalent to the evolution equations in adjoint form

$$\frac{\partial}{\partial\lambda}\begin{bmatrix} g \\ \pi \end{bmatrix} = -\mathbb{J}\circ D\Phi(g,\pi)^{*}\begin{pmatrix} N \\ X \end{pmatrix}; \quad \mathbb{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

together with the constraints

$$\Phi(g,\pi) = 0$$

where $\Phi(g,\pi) = (\mathcal{H}(g,\pi), \mathcal{J}(g,\pi))$ is the super energy-momentum. This quantity Φ is the Noether conserved quantity generated by the group $\mathcal{D}$ of diffeomorphisms of spacetime. (For asymptotically flat spacetimes, only diffeomorphisms that are spatially asymptotic to the identity are needed to generate Φ. As in [48], the Lorentz group at infinity generates the total energy momentum tensor for the spacetime; see [14]).

This machinery may now be used as a tool to investigate the structure of the space of solutions of Einstein's equations.

Let V be a fixed four manifold and let $\mathcal{E}$ be the set of all globally hyperbolic Lorentz metrics $g = {}^{(4)}g$ that satisfy the vacuum Einstein equations $\mathrm{Ein}(g) = 0$ on V (plus some additional technical smoothness conditions). Let $g_0 \in \mathcal{E}$ be a given solution. We ask: what is the structure of $\mathcal{E}$ in the neighbourhood of g_0?

There are two basic reasons why this question is asked. First of all, it is relevant to the problem of finding solutions to the Einstein equations in the form of a perturbation series:

$$g(\lambda) = g_0 + \lambda h_1 + \frac{\lambda^2}{2} h_2 + \ldots$$

where λ is a small parameter. If $g(\lambda)$ is to solve $\mathrm{Ein}(g(\lambda)) = 0$ identically in λ then clearly h_1 must satisfy the *linearized Einstein equations:*

$$D\mathrm{Ein}(g)\cdot h_1 = 0$$

where $D\mathrm{Ein}(g)$ is the derivative of the mapping $g \longrightarrow \mathrm{Ein}(g)$. For such a perturbation series to be possible, is it sufficient that h_1 satisfy the linearized Einstein equations? i.e. is h_1 necessarily a direction of *linearization stability?* We shall see that in general the answer is no, unless additional conditions hold. The second reason why the structure of $\mathcal{E}$ is of interest is in the problem of quantization of the Einstein equations. Whether one quantizes by means of direct phase space techniques (due to Dirac, Segal, Souriau and Kostant in various forms) or by Feynman path integrals, there will be difficulties near places where the space of classical solutions is such that the linearized theory is *not* a good approximation to the nonlinear theory.

The dynamical formulation mentioned above is crucial to the analysis of this problem. Indeed, the essence of the problem reduces to the study of structure of the space of solutions of the constraint equations $\Phi(g,\pi) = 0$.

The final answer to these questions is this: $\mathcal{E}$ has a conical or quadratic singularity at g_0 if and only if there is a non-trivial Killing field for g_0 that belongs to the gauge group generating $\Phi = 0$ (thus, the flat metric on $T^3 \times \mathbb{R}$ has such Killing fields, but the Minkowski metric has none.) When $\mathcal{E}$ has such a singularity, we speak of a bifurcation in the space of solutions. When $\mathcal{E}$ has no singularity, the symplectic form induced on $\mathcal{E}$ has a kernel that equals the orbits of the gauge group, so $\mathcal{E}/\mathcal{D}$ is a smooth symplectic manifold. (See Theorem 3 below).

2b. Yang-Mills Equations

There is a similar situation for gauge field theories of Yang-Mills type, possibly coupled to gravity. The final situation here is as follows. The space of solutions is a smooth manifold near solutions with no gauge symmetries and this space, modulo the gauge group, is a smooth symplectic manifold. Near solutions with a symmetry, the space of solutions has a conical singularity.

2c. History and References

The historical circumstances leading up to statements of this type are as follows:

(a) Brill and Deser [10] considered perturbations of the flat metric on $T^3 \times \mathbb{R}$ and discovered the first example of trouble in perturbation theory. They found, by going to a second order perturbation analysis, that they had to readjust the first order perturbations in order to avoid inconsistencies at second order. This was the first hint of a conical structure for $\mathcal{E}$ near solutions with symmetry.

(b) Fischer and Marsden [23] found general sufficient conditions for $\mathcal{E}$ to be a manifold in terms of the Cauchy data for vacuum spacetimes and coined the term "linearization stability". Related results were proved by O'Murchadha and York [46].

(c) Choquet-Bruhat and Deser [12,13] proved that $\mathcal{E}$ is a manifold near Minkowski space. (This was later improved by Choquet-Bruhat, Fischer and Marsden [14]).

(d) An abstract theory for systems with constraints was developed (and applied to a number of examples, including relativity) by Marsden and Weinstein [38]. They studied the general problem of the structure of the level sets (and in particular the zero sets) of conserved quantities, i.e. momentum maps, associated with a gauge or symmetry group and proved that the quotient of these level sets by the gauge group is a symplectic manifold near nonsingular (i.e. non-symmetric) points. This theory will be briefly described below with further indications of how it fits into the general scheme of relativistic field theories with constraints.

(e) Moncrief [40] showed that the sufficient conditions derived by Fischer and Marsden for the compact case where equivalent to the requirement that (V, g_0) have no Killing fields. This then led to the link between symmetries and bifurcations.

(f) Moncrief [41] discovered the general splitting of gravitational perturbations generalizing Deser's decomposition [19]. The further generalization to momentum maps (general Noether currents) was found by Arms, Fischer and Marsden [4] in the context of [38]. This then applies to other examples such as gauge theory and also gives York's decomposition [56] as special cases.

(g) D'Eath [18] obtained the basic linearization stability results for Robertson-Walker universes.

(h) Moncrief [42] discovered the spacetime significance of the second order conditions that arise when one has a Killing field and identified them with conserved quantities of Taub [54]. (Arms and Marsden [5] showed that the second order conditions for compact spacelike hypersurfaces are nontrivial conditions.)

(i) A Hamiltonian formalism for pure gauge theories of Yang-Mills type was well-known by about 1975; see [31,17,43] and references therein. This implied that the abstract results in [38] on the space of solutions can be applied directly as is explained in §3 below (once the ellipticity of the adjoint of the derivative of the constraint map is known; this simple calculation was noted in [43]). Similar facts for the pure Yang-Mills case were obtained independently by Segal [50,51] and Garcia [28].

(j) Case (i) deals with points where the space of solutions is nonsingular. The singular case was studied by Moncrief [43]. A complete proof that the singularities are conical was given by Arms [3].

(k) Coll [16] and Arms [1] carried out a study of both the singular and nonsingular points for the Einstein-Maxwell equations. In [2] the general coupled Einstein-Yang-Mills system was studied.

(l) Moncrief [44] investigated the quantum analogues of linearization stabilities. Using $T^3 \times \mathbb{R}$, he shows that unless such conditions are imposed, the correspondence principle is violated.

(m) For general relativity a detailed description of the conical singularity in $\mathcal{E}$ near a spacetime with symmetries is due to Fischer, Marsden and Moncrief [26] for one Killing field and to Arms, Marsden and Moncrief [7] in the general case.

(n) An abstraction of the results in the singular case to the general context of [38] was obtained by Arms, Marsden and Moncrief [6]. They showed quite generally that zero sets of momentum maps have conical singularities near a point with symmetry.

(o) Pilati [47] developed a Hamiltonian formalism for supergravity. This is used by Bao [9] to study the space of solutions.

3. Spaces of Solutions Near Regular Points.

To study the space of solutions of a relativistic field theory and its symplectic structure, one can carry out the following steps:

1. A "3+1 procedure" of Dirac is carried out. A symplectic manifold for the dynamics is found and the constraint equations Φ (fields $= \phi$, conjugate momentum $= \pi_\phi$) $= 0$ are isolated.
2. The constraints Φ are identified with the momentum map J for the action of an appropriate gauge group; i.e. one proves that $\Phi = J$.
3. One checks that $DJ^* = D\Phi^*$ is elliptic (in the sense of Douglis and Nirenberg for mixed systems).
4. Invoke [38] near generic (regular) points.
5. Invoke [6] near singular points; i.e. solutions with gauge symmetry.

Let us comment a little further on points 1 to 4. Step 1 is the classical Dirac procedure; we have already referred to [31] and [30] for it.

So far, Step 2 has been checked by hand for each example. The general philosophy that the constraint set can be identified with the zero set of a momentum mapping seems to be true in a remarkably large number of cases. Another example is the Einstein-Dirac equations; see [45]. Several people (Gotay, Isenberg, Marsden, Sniatycki and Yasskin) are currently investigating general contexts in which this can be proved.

Step 3 is generally a simple calculation. However, it is essential so one can justify the splitting theorems of Moncrief. This abstract theorem (see [4]) generalizes the usual decompositions of gravitational perturbations and decompositions of the Maxwell field etc. It is analogous to a Hodge-type decomposition in a symplectic context and is stated below.

Next we recall a few of the features of Step 4. To do this, we first need a bit of notation. Let M be a given manifold (possibly infinite dimensional) and let a Lie group G act on M. In examples, G will be infinite dimensional, such as the group of diffeomorphisms of a manifold or bundle automorphisms. (The proper sense in which these are Lie groups is discussed in [20]). Associated to each element ξ in the Lie algebra $\mathfrak{g}$ of G, we have a vector field ξ_M naturally induced on M. We shall denote the action by $\Phi : G \times M \longrightarrow M$ and we shall write $\Phi_g : M \longrightarrow M$ for the transformation of M associated with the group element $g \in G$. Thus

$$\xi_M(x) = \frac{d}{dt} \Phi_{\exp(t\xi)}(x) \Big|_{t=0}.$$

Now let (P,ω) be a symplectic manifold, so ω is a closed (weakly) non-degenerate two-form on P and let Φ be an action of a Lie group G on P. Assume the action is symplectic: i.e. $\Phi_g^{*}\omega = \omega$ for all $g \in G$. A *momentum mapping* is a smooth mapping $J : P \longrightarrow \mathfrak{g}^{*}$ such that

$$\langle dJ(x) \cdot v_x , \xi \rangle = \omega_x (\xi_p(x), v \;)$$

for all $\xi \in \mathfrak{g}$, $v_x \in T_xP$ where $dJ(x)$ is the derivative of J at x, regarded as a linear map of T_xP to $\mathfrak{g}^{*}$ and $\langle \, , \rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^{*}$.

A momentum map is Ad^{*}*-equivariant* when the following diagram commutes for each $g \in G$:

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "J"'] & P \arrow[d, "J"] \\ \mathfrak{g}^{*} \arrow[r, "\mathrm{Ad}^{*}_{g^{-1}}"'] & \mathfrak{g}^{*} \end{tikzcd}$$

where Ad^{*}_{g-1} denotes the co-adjoint action of G on $\mathfrak{g}^{*}$. If J is Ad^{*} equivariant, we call (P,ω,G,J) a *Hamiltonian G-space*.

Momentum maps represent the (Noether) conserved quantities associated with symmetry groups acting on phase space. This topic is of course a very old one, but it is only with more recent work of Souriau and Kostant that a deeper understanding has been achieved.

Let S_{x_0} = (the component of the identity of) $\{ g \in G | gx_0 = x_0 \}$, called the *symmetry group* of x_0. Its Lie algebra is denoted $\mathfrak{s}_x$, so

$$\mathfrak{s}_{x_0} = \{ \xi \in \mathfrak{g} \mid \xi_p(x_0) = 0 \} .$$

Let (P,ω,G,J) be a Hamiltonian G-space. If $x_0 \in P$, $\mu_0 = J(x_0)$ and if

$$dJ(x_0) \;:\; T_xP \longrightarrow \mathfrak{g}^*$$

is surjective (with split kernel), then locally $J^{-1}(\mu_0)$ is a manifold and $\{J^{-1}(\mu) \mid \mu \in \mathfrak{g}^*\}$ forms a regular local foliation of a neighbourhood of x_0. Thus, when $dJ(x_0)$ fails to be surjective, the set of solutions of $J(x) = 0$ could fail to be a manifold.

Theorem 1. $dJ(x_0)$ *is surjective if and only if* $\dim S_{x_0} = 0$; *i.e.* $\mathfrak{s}_{x_0} = \{0\}$.

Proof. $dJ(x_0)$ fails to be surjective if there is a $\xi \neq 0$ such that $\langle dJ(x_0)\cdot v_{x_0}, \xi \rangle = 0$ for all $v_{x_0} \in T_{x_0}P$. From the definition of momentum map, this is equivalent to $\omega_{x_0}(\xi_P(x_0), v_{x_0}) = 0$ for all v_{x_0}. Since ω_{x_0} is non-degenerate, this is, in turn equivalent to $\xi_P(x_0) = 0$; i.e. $\mathfrak{s}_{x_0} \neq 0$. ∎

This theorem assumes implicitly that there is a splitting

$$\mathfrak{g} = \text{Range } dJ(x_0) \oplus \text{Kernel } dJ(x_0)^* .$$

In the finite dimensional case this is automatic. In the infinite dimensional case it holds if $dJ(x_0)^*$ is an elliptic operator. In this case one also has the splitting

$$T_{x_0}P = \text{Ker } dJ(x_0) \oplus \text{Range } dJ(x_0)^*$$

These splittings are usually called the Fredholm alternative.

A corollary of theorem 1 is that $J^{-1}(\mu)$ is a smooth manifold near points x_0 with no symmetries.

Theorem 2. *The kernel of the symplectic form restricted to* $\ker dJ(x_0)$ *equals the tangent space to the* G_μ *orbit of* x_0 *at* x_0. *Here* $G_\mu = \{ g \in G \mid Ad^*_{g^{-1}}\mu = \mu \}$ *and* $\mu = J(x_0)$.
Thus, near those points with no symmetry, $P_\mu = J^{-1}(\mu) / G_\mu$ *is a smooth symplectic manifold.*

We call P_μ the *reduced* symplectic manifold.
This result is proved in [38], but it also follows from Moncrief's decomposition which, for $\mu = 0$ reads

$$T_{x_0} P \;=\; \text{Range } dJ(x_0)^* \oplus \text{Ker } dJ(x_0) \cap \text{Ker}(dJ(x_0) \circ \mathbb{J}) \oplus \text{Range } \mathbb{J} \circ dJ(x_0)^* \quad .$$

where $\mathbb{J}$ is a complex structure associated with the symplectic form. The middle summand represents the tangent space to P_μ. The proof of Moncrief's decomposition is conveniently available in a number of places, such as [24,25] and [37].

As has already been indicated, for relativistic field theories, the four dimensional equations usually split into hyperbolic evolution equations and the constraint equations $J = 0$. If the gauge group includes time translations, the evolution equations take the abstract form

$$\frac{d}{d\lambda} x(\lambda) \;=\; - \mathbb{J} \circ DJ(x(\lambda))^* \xi$$

where $\xi(\lambda) \in \mathfrak{g}$ represents a gauge choice. The space of solutions is thus represented by the set $J = 0$ in Cauchy-data space.

The symplectic structure one gets on the space of solutions by this procedure coincides with the one obtained by direct four dimensional methods (although this is not established in complete generality, it can be checked directly for a class of examples that includes all of those of interest to us).

4. A Simple Example: Electromagnetism.*

We now give a simple example of how the reduction procedure in the previous section works. We give it for electromagnetism for simplicity; the construction easily generalizes to Yang-Mills fields.

The four dimensional set-up consists of the usual Kaluza-Klein formalism. One has a circle bundle over spacetime whose connections represent electromagnetic potentials. A 3+1 analysis gives a circle bundle $\pi : B \longrightarrow M$ over a three manifold M representing a spacelike hypersurface.

Let CM be the bundle over M whose sections A are connections for the bundle B. The bundle CM is, in a canonical way, a symplectic manifold which is constructed via

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* The point of view developed in this example was obtained jointly with Alan Weinstein.

reduction as follows (cf. [55]): the group S^1 acts on B and hence on T^*B. It produces a momentum map $J : T^*B \longrightarrow \mathbb{R}$. The reduced manifold $J^{-1}(1) / S^1$ is then CM . (The choice of $1 \in \mathbb{R}$ represents a normalization for a unit charge).

Let $\mathcal{O}\!\iota$ denote all sections of CM and let $\mathcal{B}$ denote the group of all automorphisms of the bundle B. Then via reduction, $\mathcal{B}$ acts on $\mathcal{O}\!\iota$.

Elements of $T^*\mathcal{O}\!\iota$ represent pairs (A-E), where A is the potential and E is the electric field. We put on $T^*\mathcal{O}\!\iota$ the canonical symplectic structure.

Maxwell's vacuum equations in terms of A and E may be summarized as

1. Hamiltonian evolution equations in $T^*\mathcal{O}\!\iota$ for the Hamiltonian

$$H = \frac{1}{2} \int_M \left[E^2 + (dA)^2 \right] dx$$

and 2. the constraint equation $J = \operatorname{div} E = 0$.

Here J is the momentum map for the action of $\mathcal{B}$ on $T^*\mathcal{O}\!\iota$. This is a straightforward calculation. (For sources, use $J = \rho$ or, better, couple Maxwell's equations to a source and the full momentum map will be $\operatorname{div} E - \rho$.)

How are Maxwell's vacuum equations in terms of E and B obtained? One merely reduces $T^*\mathcal{O}\!\iota$ by the gauge group $\mathcal{B}$ at the value 0; i.e. we form the symplectic manifold $J^{-1}(0) / \mathcal{B}$. If M is simply connected, say $\mathbb{R}^3$, the reduced space is isomorphic to the space of pairs (B,E) where B and E are divergence free. By Theorem 2 above, this reduced space is naturally a symplectic manifold. The Poisson bracket on it may be computed to be

$$\{F,G\} = \int \left[\frac{\delta F}{\delta E} \operatorname{curl} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \operatorname{curl} \frac{\delta F}{\delta B} \right] dx$$

where F and G are real valued functions of E and B and the functional derivatives are defined in terms of the Frechet derivative by

$$DF(E,B) \cdot (E',B') = \int \left[\frac{\delta F}{\delta E} \cdot E' + \frac{\delta F}{\delta B} \cdot B' \right] dx .$$

The usual decompositions of electromagnetic fields are seen to be a special case of Moncrief's decomposition.

This example is linear so the spaces of solutions are always manifolds. However it does demonstrate nicely how the constraint equations are the zero set of a momentum map.

For other Yang-Mills fields however, the space of solutions is not a manifold, as was already pointed out in [43]. We will briefly discuss the singular case next.

5. Spaces of Solutions near Singular Points.

In general the space of solutions of a nonlinear relativistic field theory with constraints will have singularities at solutions with symmetries. As we have pointed out already, this was first hinted at in relativity by Brill and Deser [10]. For both relativity and Yang-Mills fields these singularities are known to be conical. (See the references in §2). This is especially surprising for relativity in view of the complexity of the field equations. However, from [6] there is good reason to think that this is fairly general, independent of how badly nonlinear the field theory is. On the other hand, it requires a somewhat special and complex argument for relativity.

For vacuum gravity, let us state one of the main results in the cosmological case: suppose (V, g_0) is a vacuum spacetime that has a *compact* spacelike hypersurface $M \subset V$. (Actually we also require the existence of at least one of constant mean curvature for technical reasons). Let S_{g_0} be the Lie group of isometries of g_0 and let k be its dimension.

Theorem 3.

1. ([23,40]) $k = 0$, then $\mathcal{E}$ is a smooth manifold in a neighbourhood of g_0 with tangent space at g_0 given by the solutions of the linearized Einstein equations. The symplectic form inherited naturally from $T^*\mathcal{M}$ has kernel on $\mathcal{E}$ equal to the infinitesimal gauge transformations, so the space $\mathcal{E}/\mathcal{D}$ is a symplectic manifold near such points.⁺

2. ([42,26,6,7]) If $k > 0$ then $\mathcal{E}$ is *not* a smooth manifold at g_0. A solution h_1 of the linearized equations is tangent to a curve in $\mathcal{E}$ if and only if h_1 is such that Taub conserved quantities vanish; i.e. for every Killing field X for g_0,

$$\int_M X \cdot \left[D^2 \mathrm{Ein}(g_0) \cdot (h_1, h_1) \right] \cdot Z \, dx = 0$$

where Z is the unit normal to the hypersurface M and "$\cdot$" denotes contraction with respect to the metric g_0.

⁺ The proof of a technical, but important item, namely that near points with no symmetry, $\mathcal{E}/\mathcal{D}$ is a manifold has not yet appeared in the literature. It will appear in a forthcoming publication of Isenberg and Marsden.

All explicitly known solutions possess symmetries, so while 1. is "generic", 2. is what occurs in examples. This theorem gives a complete answer to the perturbation question: a perturbation series is possible if and only if all the Taub quantities vanish. Thus, the second order conditions of Taub tell us the tangents to the conical singularity. There is a similar theorem for Yang-Mills fields [6,3].

Let us give a brief abstract indication of why such second order conditions should come in. Suppose X and Y are Banach spaces and $F : X \longrightarrow Y$ is a smooth map. In our examples, F will be a momentum map. Suppose $F(X_0) = 0$ and $x(\lambda)$ is a curve with $x(0) = x_0$ and $F(x(\lambda)) \equiv 0$. Let $h_1 = x'(0)$ so by the chain rule $DF(x_0) \cdot h_1 = 0$. Now suppose $DF(x_0)$ is not surjective and in fact suppose there is a linear functional $\ell \in Y^*$ orthogonal to its range: $\langle \ell, DF(x_0) \cdot u \rangle = 0$ for all $u \in X$. (Recall from Theorem 1 that dJ fails to be surjective at points with symmetry.) By differentiating $F(x(\lambda)) = 0$ twice at $\lambda = 0$, we get

$$D^2F(x_0) \cdot (h_1, h_1) + DF(x_0) \cdot x''(0) = 0 .$$

Applying ℓ gives

$$\langle \ell , D^2F(x_0) \cdot (h_1, h_1) \rangle = 0$$

which are necessary second order conditions that must be satisfied by h_1.

It is by this general method that one arrives at the Taub conditions. The issue of whether or not these conditions are sufficient is much deeper requiring extensive analysis and bifurcation theory (for $k = 1$, the Morse lemma is used, while for $k > 1$ the Kuranishi deformation theory is needed; see [35,8,6]).

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